Highly entangled states:
A totally new kind of quantum materials
and an unification of light and electrons

Highly entangled states: A totally new kind of quantum materials and an unification of light and electrons


Postdoc at Institute for Advanced Study, Tsinghua University
2-year postdoctoral Fellows at $20,000/year
3-year C. N. Yang Junior Fellows at $30,000/year
Please apply (Baidu “IASTU” http://www.castu.tsinghua.edu.cn)
In primary school, we learned ...

there are four states of matter:

Solid

Liquid

Gas

Plasma
In university, we learned...

different phases = different symmetry breaking

→ Landau symmetry breaking theory

Group theory: From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D
In graduate study after 1980’s, we learned ...

there are phases beyond symmetry-breaking:

\[ E_y = R_{HJx}, \quad R_H = \frac{p}{q} \frac{h}{e^2} \]

- 2D electron gas in magnetic field has many quantum Hall (QH) states


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In graduate study after 1980’s, we learned ...

there are phases beyond symmetry-breaking:

\[ E_y = R_H j_x, \quad R_H = \frac{p}{q} \frac{h}{e^2} \]

- 2D electron gas in magnetic field has many quantum Hall (QH) states that all have the same symmetry.
- Different QH states cannot be described by symmetry breaking theory.
- We call the new order topological order \cite{Wen89}


Highly entangled states: A totally new kind of quantum materials
What is topological order?

- Topological order is a property or a pattern in the ground state wave function
  \[ \Phi(x_1, x_2, ..., x_N), \quad N \sim 10^{10} - 10^{23} \]
  But how to see a pattern in a wave function that we cannot even write down?
- Symmetry breaking order is also a pattern in the ground state wave function, where we examine if the wave function is invariant under symmetry operation \( U \) or not:
  \[ U[\Phi(x_1, x_2, ..., x_N)] = \Phi(x_1, x_2, ..., x_N) \]
  \( \rightarrow \) pattern of symmetry breaking.
- Use dancing picture to understand the pattern of topological order and pattern of symmetry breaking order.
Symmetry breaking orders through pictures

Ferromagnet

Anti-ferromagnet

Superfluid of bosons

Superconductor of fermions

- every spin/particle is doing its own dancing,
- every spin/particle is doing the same dancing $\rightarrow$ Long-range order

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Topological orders through pictures

- **Global dance:**
  All spins/particles dance following a local dancing “rules”
  → The spins/particles dance collectively
  → a global dancing pattern.

FQH state

String liquid (spin liquid)
Local dancing rule $\rightarrow$ global dancing pattern

- Local dancing rules of a FQH liquid:
  1. every electron dances around clock-wise ($\Phi_{\text{FQH}}$ only depends on $z = x + iy$)
  2. takes exactly three steps to go around any others ($\Phi_{\text{FQH}}$’s phase change $6\pi$)

$\rightarrow$ Global dancing pattern $\Phi_{\text{FQH}}(\{z_1, ..., z_N\}) = \prod(z_i - z_j)^3$

- Local dancing rules are enforce by the Hamiltonian to lower the ground state energy.
Local dancing rule $\rightarrow$ global dancing pattern

- Local dancing rules of a string liquid:
  1. Dance while holding hands (no open ends)
  2. $\Phi_{\text{str}}(\text{ring}) = \Phi_{\text{str}}(\text{wall})$, $\Phi_{\text{str}}(\text{wall} \rightarrow \text{wall}) = \Phi_{\text{str}}(\text{wall})$

$\rightarrow$ Global dancing pattern $\Phi_{\text{str}}(\text{loops}) = 1$


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Local dancing rule → global dancing pattern

• Local dancing rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) \( \Phi_{\text{str}}(\text{closed}) = \Phi_{\text{str}}(\text{open}) \), \( \Phi_{\text{str}}(\text{open}) = \Phi_{\text{str}}(\text{closed}) \)
  → Global dancing pattern \( \Phi_{\text{str}}(\text{loops}) = 1 \)

• Local dancing rules of another string liquid:
  (1) Dance while holding hands (no open ends)
  (2) \( \Phi_{\text{str}}(\text{closed}) = \Phi_{\text{str}}(\text{open}) \), \( \Phi_{\text{str}}(\text{open}) = -\Phi_{\text{str}}(\text{closed}) \)
  → Global dancing pattern \( \Phi_{\text{str}}(\text{loops}) = (-)^{\# \text{ of loops}} \)

• Two string-net condensations → two topological orders


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What is the significance of topological order?

Globel danicing pattern is a nice picture for topological order. But does it mean anything?
Does topological order have any experiemntal significance?
Does topological order have any new experiemntal properties, that is different from any symmetry breaking order?

How to measure/study topological order in experiemnts?
What is the significance of topological order?

Globel danicing pattern is a nice picture for topological order. But does it mean anything? Does topological order have any experimental significance? Does topological order have any new experimental properties, that is different from any symmetry breaking order?

**How to measure/study topological order in experiments?**

- Topological orders produce **new kind of waves** (collective excitations above the topo. ordered ground states). → *change our view of universe*

- The defects of topological order carry **fractional statistics** (including non-Abelian statistics) and **fractional charges** (if there is symmetry). → *a medium for topological quantum memory and computations.*

- Some topological orders have topologically protected **gapless boundary excitations** → *perfect conducting surfaces despite the insulating bulk.*
Principle of emergence: from order to physical properties

**Different orders → different wave equations for the deformations of order → different physical properties.**

- Atoms in superfluid have a random distribution
  → cannot resist shear deformations (which do nothing)
  → liquids do not have shapes

Wave Eq. → Euler Eq.
\[ \partial_t^2 \rho - \partial_x^2 \rho = 0 \quad \text{One longitudinal mode} \]

- Atoms in solid have an ordered lattice distribution
  → can resist shear deformations
  → solids have shapes

Wave Eq. → elastic Eq.
\[ \partial_t^2 u^i - C^{ijkl} \partial_{x^j} \partial_{x^k} u^l = 0 \]

One longitudinal mode and two transverse modes

Highly entangled states: A totally new kind of quantum materials

Origin of photons, gluons, electrons, quarks, etc

- Do all waves and wave equations emerge from some orders?

**Wave equations for elementary particles**

- Maxwell equation $\rightarrow$ Photons
  \[ \partial \times E + \partial_t B = \partial \times B - \partial_t E = \partial \cdot E = \partial \cdot B = 0 \]

- Yang-Mills equation $\rightarrow$ Gluons
  \[ \partial^\mu F^a_{\mu\nu} + f^{abc} A^b_{\mu} F^c_{\mu\nu} = 0 \]

- Dirac equation $\rightarrow$ Electrons/quarks
  \[ [\partial_\mu \gamma^\mu + m] \psi = 0 \]

What orders produce the above waves? What are the origins of light (gauge bosons) and electrons (fermions)?
Elementary or emergent?

- But none of the symmetry breaking orders can produce:
  - electromagnetic wave satisfying the Maxwell equation
  - gluon wave satisfying the Yang-Mills equation
  - electron wave satisfying the Dirac equation.

We have seen that topological order corresponds to new states of quantum matter, such as FQH states and spin-liquid states.

Can topological order produce the Maxwell equation, Yang-Mills equation, and the Dirac equation?

Yes

A particular topologically ordered state, string-net liquid, provides a unified origin of light, electrons, quarks, gluons, ... ...


Highly entangled states: A totally new kind of quantum matter
Elementary or emergent?

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- Can topological order produce the Maxwell equation, Yang-Mills equation, and the Dirac equation?
  
  **Yes**

- A particular topologically ordered state, string-net liquid, provide a unified origin of light, electrons, quarks, gluons, ... ...
Topological order (closed strings) → emergence of electromagnetic waves (photons)

- Wave in superfluid state $|\Phi_{SF}\rangle = \sum_{\text{all position conf.}} |\phi\rangle$
  
  density fluctuations:
  Euler eq.: $\partial_t^2 \rho - \partial_x^2 \rho = 0$
  → Longitudinal wave

- Wave in closed-string liquid $|\Phi_{\text{string}}\rangle = \sum_{\text{closed strings}} |\phi\rangle$

String density $E(x)$ fluctuations → waves in string condensed state.
Strings have no ends → $\partial \cdot E = 0$ → only two transverse modes.
Equation of motion for string density → Maxwell equation:
$\dot{E} - \partial \times B = \dot{B} + \partial \times E = \partial \cdot B = \partial \cdot E = 0$. ($E$ electric field)
Topological order (string nets) $\rightarrow$ Emergence of Yang-Mills theory (gluons)

- If string has different types and can branch $\rightarrow$ string-net liquid $\rightarrow$ Yang-Mills theory
- Different ways that strings join $\rightarrow$ different gauge groups

Closed strings $\rightarrow$ Maxwell gauge theory
String-nets $\rightarrow$ Yang-Mills gauge theory

Topological order →
Emergence of electrons

- In string condensed states, the ends of string behave like point particles
  - with quantized (gauge) charges
  - with Fermi statistics

Levin-Wen 2003

- **String-net/topological-order provides a way to unify gauge interactions and Fermi statistics in 3D**
Emergence of fractional spin/statistics

• Why electron carry spin-1/2 and Fermi statistics?

• $\Phi_{\text{str}}(\text{\rotatebox{180}{\includegraphics{string_liquid_imaginary.png}}}) = 1$ string liquid $\Phi_{\text{str}}(\text{\includegraphics{string_liquid_left.png}}) = \Phi_{\text{str}}(\text{\includegraphics{string_liquid_right.png}})$

360° rotation: $\updownarrow \rightarrow \curlycirc $ and $\updownarrow = \curlycirc \rightarrow \updownarrow$

$\updownarrow + \curlycirc$ has a spin 0 mod 1. $\updownarrow - \curlycirc$ has a spin 1/2 mod 1.
Emergence of fractional spin/statistics

• Why electron carry spin-1/2 and Fermi statistics?

• $\Phi_{\text{str}} (\square\square\square) = 1$ string liquid $\Phi_{\text{str}} (\square\square\square) = \Phi_{\text{str}} (\square\square\square)$

360° rotation: $\uparrow \rightarrow \bigcirc$ and $\bigcirc = \bigcirc \rightarrow \uparrow$

$\uparrow + \bigcirc$ has a spin 0 mod 1. $\uparrow - \bigcirc$ has a spin 1/2 mod 1.

• $\Phi_{\text{str}} (\square\square\square) = (-)^{\# \text{ of loops}}$ string liquid $\Phi_{\text{str}} (\square\square\square) = -\Phi_{\text{str}} (\square\square\square)$

360° rotation: $\uparrow \rightarrow \bigcirc$ and $\bigcirc = -\bigcirc \rightarrow -\uparrow$

$\uparrow + i\bigcirc$ has a spin $-1/4$ mod 1. $\uparrow - i\bigcirc$ has a spin 1/4 mod 1.
Spin-statistics theorem

- (a) → (b) = exchange two string-ends.
- (d) → (e) = $360^\circ$ rotation of a string-end.
- Amplitude (a) = Amplitude (e)
- Exchange two string-ends plus a $360^\circ$ rotation of one of the string-end generate no phase.

→ Spin-statistics theorem
To define a physical concept, such as symmetry-breaking order or topological order, is to design a probe to measure it.

For example,
- crystal order is defined/probed by X-ray diffraction:
Symmetry-breaking orders through experiments

<table>
<thead>
<tr>
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<th>Experiment</th>
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<tbody>
<tr>
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<td>Neutron scattering</td>
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<tr>
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<td>Zero-resistance &amp; Meissner effect</td>
</tr>
<tr>
<td>Topological order</td>
<td>???</td>
</tr>
</tbody>
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- All the above probes are linear responses. But topological order cannot be probed/defined through linear responses.
Topological orders through experiments (1990)

*Topological order can be defined “experimentally” through two unusual topological probes (at least in 2D)*

1. **Topology-dependent ground state degeneracy** $D_g$  
   
   - $g=0$: 
     - Deg. = 1
   - $g=1$: 
     - Deg. = $D_1$
   - $g=2$: 
     - Deg. = $D_2$

2. **Non-Abelian geometric’s phases** of the degenerate ground state from deforming the torus:
   
   - Shear deformation $T$: $|\psi_\alpha\rangle \rightarrow |\psi'_\alpha\rangle = T_{\alpha\beta}|\psi_\beta\rangle$
   
   - $90^\circ$ rotation $S$: $|\psi_\alpha\rangle \rightarrow |\psi''_\alpha\rangle = S_{\alpha\beta}|\psi_\beta\rangle$

- $T, S$, define topological order “experimentally”.
- $T, S$ is a *universal probe* for any 2D topological orders, just like X-ray is a universal probe for any crystal orders.
Symmetry-breaking/topological orders through experiments

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<td>Topological order (Global dancing pattern)</td>
<td>Topological degeneracy, non-Abelian geometric phase</td>
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- The linear-response probe **Zero-resistance** and **Meissner effect** define **superconducting order**. Treating the EM fields as non-dynamical fields

- The topological probe **Topological degeneracy** and **non-Abelian geometric phases** $T, S$ define a completely new class of order – **topologically order**.

- $T, S$ determines the quasiparticle statistics. Keski-Vakkuri & Wen 93; Zhang-Grover-Turner-Oshikawa-Vishwanath 12; Cincio-Vidal 12

Highly entangled states: A totally new kind of quantum materials
What is the microscopic picture of topological order?

![Diagrams showing different states of topological order](image)

Deg. = D
Deg. = D
Deg. = D

represent an experimental definition of topological order.

- But what is the microscopic understanding of topological order?
- Zero-resistance and Meissner effect → experimental definition of superconducting order.
- It took 40 years to gain a microscopic picture of superconducting order: **electron-pair condensation**

Bardeen-Cooper-Schrieffer 57
What is the microscopic picture of topological order?

- But what is the microscopic understanding of topological order?
- Zero-resistance and Meissner effect → experimental definition of superconducting order.
- It took 40 years to gain a microscopic picture of superconducting order: \textit{electron-pair condensation} Bardeen-Cooper-Schrieffer 57

- It took 20 years to gain a microscopic understanding of topological order: \textit{long-range entanglements} Chen-Gu-Wen 10 (defined by local unitary trans. and motivated by topological entanglement entropy). Kitaev-Preskill 06, Levin-Wen 06

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Quantum entanglements through examples

- $|↑⟩ \otimes |↓⟩ = \text{direct-product state} \rightarrow \text{unentangled (classical)}$


Highly entangled states: A totally new kind of quantum materials
Quantum entanglements through examples

- $|↑⟩⊗|↓⟩ = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
- $|↑⟩⊗|↓⟩ + |↓⟩⊗|↑⟩ \rightarrow \text{entangled (quantum)}$
Quantum entanglements through examples

- $|↑⟩\otimes|↓⟩ = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
- $|↑⟩\otimes|↓⟩ + |↓⟩\otimes|↑⟩ \rightarrow \text{entangled (quantum)}$
- $|↑⟩\otimes|↑⟩ + |↓⟩\otimes|↓⟩ + |↑⟩\otimes|↓⟩ + |↓⟩\otimes|↑⟩ \rightarrow \text{more entangled}$
Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
- $|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \rightarrow \text{entangled (quantum)}$
- $|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle$
  $= (|\uparrow\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |\downarrow\rangle) = |x\rangle \otimes |x\rangle \rightarrow \text{unentangled}$

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- $\cdots \rightarrow \text{unentangled}$
Quantum entanglements through examples

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- $|\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots \rightarrow \text{unentangled}$
- $|\downarrow\rangle \uparrow\rangle \downarrow\rangle \uparrow\rangle \downarrow\rangle = (|\downarrow\rangle \uparrow\rangle \downarrow\rangle \uparrow\rangle \downarrow\rangle) \otimes (|\downarrow\rangle \uparrow\rangle \downarrow\rangle \uparrow\rangle \downarrow\rangle) \otimes \ldots \rightarrow \text{short-range entangled (SRE) entangled}$
Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
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\[
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\]
- $\begin{array}{cccccc}
\text{\text{\uparrow}} & \text{\text{\downarrow}} & \text{\text{\uparrow}} & \text{\text{\downarrow}} & \text{\text{\uparrow}} & \text{\text{\downarrow}} \\
\text{\text{\downarrow}} & \text{\text{\uparrow}} & \text{\text{\downarrow}} & \text{\text{\uparrow}} & \text{\text{\downarrow}} & \text{\text{\uparrow}}
\end{array}
= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \rightarrow \text{unentangled}

- $\begin{array}{cccccc}
\text{\text{\uparrow}} & \text{\text{\uparrow}} & \text{\text{\uparrow}} & \text{\text{\uparrow}} & \text{\text{\uparrow}} & \text{\text{\uparrow}}
\text{\text{\downarrow}} & \text{\text{\downarrow}} & \text{\text{\downarrow}} & \text{\text{\downarrow}} & \text{\text{\downarrow}} & \text{\text{\downarrow}}
\end{array}
= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \ldots \rightarrow \text{short-range entangled (SRE) entangled}
$

- Crystal order: $|\Phi_{\text{crystal}}\rangle = |\begin{array}{c}
\text{\text{\uparrow}} & \text{\text{\uparrow}} & \text{\text{\uparrow}} & \text{\text{\uparrow}} & \text{\text{\uparrow}} & \text{\text{\uparrow}} \\
\text{\text{\downarrow}} & \text{\text{\downarrow}} & \text{\text{\downarrow}} & \text{\text{\downarrow}} & \text{\text{\downarrow}} & \text{\text{\downarrow}}
\end{array}| = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \ldots$

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= \text{direct-product state} \rightarrow \text{unentangled state (classical)}
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Quantum entanglements through examples

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- $\cdots \rightarrow \text{unentangled}$
- $\cdots = (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \cdots \rightarrow \text{short-range entangled (SRE) entangled}$

- Crystal order: $|\Phi_{\text{crystal}}\rangle = |\cell\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \rightarrow \text{direct-product state} \rightarrow \text{unentangled state (classical)}$
- Particle condensation (superfluid) $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} |\cell\rangle$
Quantum entanglements through examples

- $|\uparrow\rangle \otimes |\downarrow\rangle = \text{direct-product state} \rightarrow \text{unentangled (classical)}$
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- Particle condensation (superfluid)
  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} |\ldots\rangle = (|0\rangle_{x_1} + |1\rangle_{x_1} + ..) \otimes (|0\rangle_{x_2} + |1\rangle_{x_2} + ..) \ldots = \text{direct-product state} \rightarrow \text{unentangled state (classical)}$

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To make topological order, we need to sum over different product states, but we should not sum over everything.

- Sum over a subset of the particle configurations, by first join the particles into strings, then sum over the loop states

\[ |\Phi_{\text{loop}}\rangle = \sum_{\text{all loop conf.}} |\text{loop state}\rangle \]

which is not a direct-product state and not a local deformation of direct-product states

→ non-trivial topological orders (long-range entanglements)

→ string-net condensation (string liquid): Levin-Wen 05

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Pattern of long-range entanglements $\equiv$ topological order

**For gapped systems with no symmetry:**

- According to Landau theory, no symmetry to break
  $\rightarrow$ all systems belong to one trivial phase

\[
|\text{LRE}\rangle \neq |\text{product state}\rangle = |\text{SRE}\rangle
\]

- There are long range entangled (LRE) states $\rightarrow$ many phases
- There are short range entangled (SRE) states $\rightarrow$ one phase

\[\text{local unitary transformation} \]

\[|\text{LRE}_1 \text{ LRE}_2\rangle \text{ product state} \]

\[|\text{SRE}_1 \text{ SRE}_2\rangle \text{ product state} \]

All SRE states belong to the same trivial phase

LRE states can belong to many different phases

= different patterns of long-range entanglements

= different topological orders

$\rightarrow$ A classification by tensor category theory

Levin-Wen 05, Chen-Gu-Wen 2010


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Pattern of long-range entanglements = topological order

For gapped systems with no symmetry:

• According to Landau theory, no symmetry to break → all systems belong to one trivial phase

• Thinking about entanglement: Chen-Gu-Wen 2010
  - There are long range entangled (LRE) states
  - There are short range entangled (SRE) states

\[ |\text{LRE}\rangle \neq |\text{product state}\rangle = |\text{SRE}\rangle \]
Pattern of long-range entanglements = topological order

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break → all systems belong to one trivial phase

- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are long range entangled (LRE) states → many phases
  - There are short range entangled (SRE) states → one phase

- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different patterns of long-range entanglements defined by the LU trans.
  = different topological orders
  → A classification by tensor category theory, Levin-Wen 05, Chen-Gu-Wen 2010

Highly entangled states: A totally new kind of quantum materials
Local dancing rule $\rightarrow$ global dancing pattern

- Local dancing rules of a string liquid:
  1. Dance while holding hands (no open ends)
  2. $\Phi_{\text{str}}(\text{a}) = \Phi_{\text{str}}(\text{b})$, $\Phi_{\text{str}}(\text{c} \rightarrow \text{d}) = \Phi_{\text{str}}(\text{e})$
  $\rightarrow$ Global dancing pattern $\Phi_{\text{str}}(\text{f}) = 1$

- Local dancing rules of another string liquid:
  1. Dance while holding hands (no open ends)
  2. $\Phi_{\text{str}}(\text{a}) = \Phi_{\text{str}}(\text{b})$, $\Phi_{\text{str}}(\text{c} \rightarrow \text{d}) = -\Phi_{\text{str}}(\text{e})$
  $\rightarrow$ Global dancing pattern $\Phi_{\text{str}}(\text{f}) = (-)^{\# \text{ of loops}}$

- Two string-net condensations $\rightarrow$ two topological orders [Levin-Wen 05]

Highly entangled states: A totally new kind of quantum materials and an unification of light and electrons
More general dancing rules $\rightarrow$ Tensor category theory

The local dancing rules can be described by data $d_i, N_{ijk}, F_{klm}^{ij}$:

\[
\Phi \left( \begin{array}{c}
\bullet \\
\end{array} \right) = d_i \Phi \left( \begin{array}{c}
\bullet \\
\end{array} \right)
\]

\[
\Phi \left( \begin{array}{c}
l \\
\end{array} \\
\end{array} \right) = \delta_{ij} N_{ilk} \Phi \left( \begin{array}{c}
l \\
\end{array} \right)
\]

\[
\Phi \left( \begin{array}{c}
i \\
\end{array} \\
\end{array} \right) = \sum_{n=0}^{N} F_{klm}^{ijm} \Phi \left( \begin{array}{c}
l \\
\end{array} \right)
\]

which must satisfy

\[
F_{ijk} = \frac{v_k}{v_i v_j} N_{ijk}, \quad v_i^2 = d_i
\]

\[
F_{klm}^{ijm} = F_{lijm} = F_{kim}^{lm} = F_{knlm}^{ijm} \frac{v_m v_n}{v_j v_l}
\]

\[
\sum_{n=0}^{N} F_{kln}^{mlq} F_{mns}^{jip} F_{lkr}^{jsn} = F_{qkr}^{jsn} F_{rlm}^{jip}
\]

The theory about the solutions $= \text{tensor category theory}$

$\rightarrow$ classify 2D gapped phases with no symmetry (topological order)
Short-range entanglements that break symmetry → Landau symmetry breaking phases

different phases = different symmetry breaking

From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D

Highly entangled states: A totally new kind of quantum materials and an unification of light and electrons
Short-range entanglements w/ symmetry $\rightarrow$ SPT phases

For gapped systems with a symmetry $G$ (no symmetry breaking):

- there are **LRE symmetric states** $\rightarrow$ many different phases
- there are **SRE symmetric states** $\rightarrow$ one phase

Short-range entanglements w/ symmetry $\rightarrow$ SPT phases

For gapped systems with a symmetry $G$ (no symmetry breaking):
- there are **LRE symmetric states** $\rightarrow$ many different phases
- there are **SRE symmetric states** $\rightarrow$ many different phases

We may call them **symmetry protected trivial** (SPT) phase

$\text{SY-SRE} 1$  $\text{SB-SRE} 1$

$\text{SY-SRE} 2$  $\text{SB-SRE} 2$

$\text{SY-LRE} 1$  $\text{SB-LRE} 1$

$\text{SY-LRE} 2$  $\text{SB-LRE} 2$

$\text{topological orders}$ ( ??? )

$\text{symmetry breaking}$ (group theory)
$\text{SPT phases}$ ( ??? )

$\text{topological order}$

SPT 1  SPT 2

$\times$  preserve symmetry

phase transition

no symmetry

---


Highly entangled states: A totally new kind of quantum materials
Short-range entanglements w/ symmetry → SPT phases

For gapped systems with a symmetry $G$ (no symmetry breaking):

- there are LRE symmetric states → many different phases
- there are SRE symmetric states → many different phases

We may call them symmetry protected trivial (SPT) phase

- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83
Short-range entanglements w/ symmetry → SPT phases

For gapped systems with a symmetry $G$ (no symmetry breaking):

- there are **LRE symmetric states** $\rightarrow$ many different phases
- there are **SRE symmetric states** $\rightarrow$ many different phases

We may call them **symmetry protected trivial (SPT)** phase or **symmetry protected topological (SPT)** phase

- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm.  
  - Topo. insulators w/ $U(1) \times T$ symm.: 2D Kane-Mele 05; Bernevig-Zhang 06 and 3D Moore-Balents 07; Fu-Kane-Mele 07

Highly entangled states: A totally new kind of quantum materials
Compare topological order and topological insulator

- **Topological order** describes states with *long-range entanglements*
  - The essence: long-range entanglements

- **Topological insulator** is a state with *short-range entanglements, particle number conservation, and time reversal symmetry*, which is an example of SPT phases.
  - The essence: symmetry entangled with short-range entanglements

<table>
<thead>
<tr>
<th></th>
<th>Topo. order</th>
<th>Topo. Ins.</th>
<th>Band Ins.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entanglements</td>
<td>long range</td>
<td>short range</td>
<td>short range</td>
</tr>
<tr>
<td>Fractional charges of finite-energy defects</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Fractional statistics of finite-energy defects</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Proj. non-Abelian stat. of infinite-energy defects</td>
<td>Yes (Majorana)</td>
<td>Yes (only Majorana)</td>
<td>Yes (only Majorana)</td>
</tr>
<tr>
<td>Gapless boundary</td>
<td>topo. protected</td>
<td>symm. protected</td>
<td>not protected</td>
</tr>
</tbody>
</table>
Highly entangled quantum matter: A new chapter of condensed matter physics

- Highly entangled states: A totally new kind of quantum materials and an unification of light and electrons

- Tensor category theory
- Long−Rang Entangled = Topological order
- FQH State
- Real Super−conductor
- Spin Liquid
- Gauge theory
- Fractional statistics
- Boundary excitations
- Topological degeneracy

- Quantum States of Matter (gapped)
  - Group Theory
  - Symm. Breaking
  - Crystal
  - Liquid
  - Magnet
- Short−Range Entangled
  - Group theory
  - With symmetry
  - Haldane Phase
  - Topological Insulator
- Boundary excitations
  - Anti−localization
- Insulator
  - Information storage
  - Gapless Goldstone mode

- Group cohomology theory

Topological order is the source of many wonders.